

THE GROUP G_n^2 WITH A PARITY AND WITH POINTS

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ABSTRACT. In [8] Manturov studied groups G_n^k for fixed integers n and k such that $k < n$. In particular, G_n^2 is isomorphic to the group of free braids of n -stands. In [6] Manturov and the author studied an invariant valued in free groups not only for free braids but also for free tangles, which is derived from the group G_n^2 . On the other hands, in [3] Manturov and Fedoseev studied groups Br_2^n of virtual braids with parity and groups Br_d^n of virtual braids with dots. They showed that there is homomorphism from Br_2^n to Br_d^n and proved the following statement:

if two braids with parity are equivalent as braids with dots, then they are equivalent as braids with parity.

By the statement it is deduced that a parity of the braid can be represented by a geometric object, dots on strands.

In this paper we study G_n^2 with structures, which are corresponded to parity and points on a braid, which are denoted by $G_{n,p}^2$ and $G_{n,d}^2$, respectively. In section 3, it is proved that there is a monomorphism from G_n^2 to $G_{n,p}^2$ and that there is a monomorphism from $G_{n,p}^2$ to $G_{n,d}^2$. By the homomorphism from G_n^2 to $G_{n,p}^2$, it can be deduced that a given parity of a braid has geometric representation, which is the number of points on the braid. In section 4, it can be proved that for each element β in $G_{n,d}^2$, an element in G_{n+1}^2 obtained by adding another strand by tracing points on β . That is, a parity of free braid of n -stands is represented not only by points on strands, but also by an $n + 1$ -th strand. Conversely, for a braid of $n + 1$ -strands, a braid of n -strands is obtained by deleting one strand of the braid of $n + 1$ -strand. Finally, we will simply discuss about the way to adjust the previous observations to know whether a given braid is Brunnian or not.

1. INTRODUCTION

In [8], Manturov studied groups G_n^k given by the group presentation, from which many invariants for knots and dynamical systems are derived. In [10], V.O.Manturov and I.M.Nikonov found that pure braids groups are embedded in G_n^k and constructed invariants valued in free product of cyclic groups. Tangles are significant generalization of braids and we have the following question : is it possible to generalize invariants, which are asserted, to the case of tangles. By G_n^2 analogously tangles can be studied. In [6], the invariants valued in free groups are extended to the case of free tangles. The groups G_n^2 which are simplest case of the G_n^k are known as free braid groups of n -strands, which are known as other names, see [1] and [2].

On the other hands, in [3] V.O.Manturov and D.A.Fedoseev studied virtual braids group Br_2^n with parity and virtual braids group Br_d^n with dots. They showed that there is a homomorphism h from Br_2^n to Br_d^n . And it was shown that Br_2^n is homomorphically embedded in Br_d^n by showing the following statement:

if two braids with parity are equivalent as braids with dots, then they are equivalent as braids with parity.

It means that the parity of braids has geometric meaning; the number of points on braids.

In this paper, analogously we will generalize the groups G_n^2 with structures; parity and points, which are denoted by $G_{n,p}^2$ and $G_{n,d}^2$, respectively. We are interested in the following question:

two objects are equivalent in a larger category, then are they equivalent as themselves?

Which means that a category is embedded in the larger category. For example, the following statements are related with the above problem.

- (1) if two classical knots are equivalent as virtual knots, then they are equivalent as classical knots.
- (2) if two classical braids are equivalent as virtual braids, then they are equivalent as classical braids.

In [5] the first statement is proved and the second statement is proved in [4].

As the above question, we showed the following two statements:

- (1) if two element in G_n^2 are equivalent in $G_{n,p}^2$, then they are equivalent in G_n^2 .
- (2) if two element in $G_{n,p}^2$ are equivalent in $G_{n,d}^2$, then they are equivalent in $G_{n,p}^2$.

From the above statements, the followings are deduced: Since G_n^2 is isomorphic to group of free braids with n strands, parity can be generalized to the case of free braids with respect to $G_{n,p}^2$. Moreover, parity is defined abstractly, but it can be geometrically presented by points. On the other hands point on a braid of n strands means 'linking' between $(n+1)$ -th strand and others. Moreover points on the braid of n -strands can be considered as *traces* of $n+1$ -th strand and a homomorphism from $G_{n,d}^2$ to G_{n+1}^2 can be defined by adding a new strand by 'following the traces'(points) of it. By the above observations, we can find a homomorphism from $G_{n,p}^2$ to G_{n+1}^2 and, roughly speaking, each value of parity of crossings of the braid of n -strands can be represented by the number of 'linking' between $n+1$ -strand and the other two strands, in which a crossing is. Conversely a braid with points can be obtained from a braid of $n+1$ strands by deleting one strand and placing points on places, on which deleted strand was. And a braid with parity can be obtained from a braid with points by counting the number of points. Then two braids can be distinguished by braids with parity. In section 2, we simply remind basic definitions and simple results. In section 3, we consider the following question:

two objects are equivalent in a larger category, then are they equivalent as themselves?

in the case of G_n^2 , $G_{n,p}^2$ and $G_{n,d}^2$. In section 4, relations between $G_{n,d}^2$ and G_{n+1}^2 are studied. And we will show geometric meanings and examples of braids of $n+1$ strand, which are distinguished from trivial braids by braids with parity.

2. BASIC DEFINITIONS

Firstly, we recall basic definitions and introduce simple results.

Definition 2.1. [9] Let G_n^2 be the group given by the presentation generated by $\{a_{ij} \mid \{i, j\} \subset \{1, \dots, n\}, i < j\}$ with the following relations:

- (1) $a_{ij}^2 = 1$ for all $i \neq j$,

- (2) $a_{ij}a_{kl} = a_{kl}a_{ij}$ for distinct i, j, k, l ,
- (3) $a_{ij}a_{ik}a_{jk} = a_{jk}a_{ik}a_{ij}$ for distinct i, j, k .

In [9] Manturov showed that *colored (pure) free braids* can be presented by elements in G_n^2 . In fact, the relations for G_n^2 are related with Reidemeister moves for free braids.

Definition 2.2. A *free braid diagram* is a graph inside a rectangle $\mathbb{R} \times [0, 1]$ with the following properties:

- (1) The graph vertices of valency 1 are the points $[i, 0]$ and $[i, 1]$, $i = 1, \dots, n$.
- (2) All other vertices are 4-valent vertices.
- (3) For each 4-valent vertex, edges are split by two pairs, two edges in which are called *opposite edges*. The opposite edges in such vertices are at the angle of π .
- (4) The braid strands monotonously go down.

For a free braid diagram, if $[i, 0]$ and $[i, 1]$ are included in the same strand for every $i \in \{1, \dots, n\}$, then the diagram is called a *pure free braid diagram*.

A *colored free braid diagram* is a free braid diagram such that each component is enumerated. If it is a pure free braid diagram, we may assume that the enumeration agrees with x -terms of 1-valent vertices of the strands.

Definition 2.3. A *free braid of n strand* is an equivalence class of free diagrams of n -strand braid under Artin moves for free braids in Fig. 1.

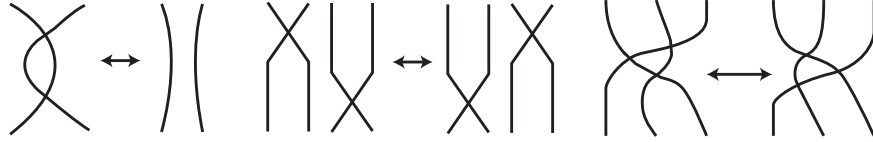


FIGURE 1. Artin moves for free braids

This moves are related with relations of G_n^2 in order. Now we construct a function from the set PB_n of colored pure free braids with n components to G_n^2 . Let $\{\epsilon_k\}_{k=1}^m \subset (0, 1)$ be a set of values such that a 4-valent vertex is placed in $\mathbb{R} \times \{\epsilon_k\}$ and $\epsilon_k < \epsilon_{k+1}$ for each $k = 1, \dots, m-1$. Without loss of generality, we assume that there is the only one vertex on $\mathbb{R} \times \{\epsilon_k\}$ for each k . Let c_k be the 4-valent vertex on $\mathbb{R} \times \{\epsilon_k\}$. Define a function ι from PB_n to G_n^2 as the following: For each vertex c_k , if it contains edges in i_k -th and j_k -th components, then give a generator $a_{i_k j_k}$ in G_n^2 . Define $\iota(\beta) = a_{i_1 j_1} \cdots a_{i_m j_m}$, see Fig. 2.

Proposition 2.4. [9] A function ι from PB_n to G_n^2 is well-defined.

From the discovery of parity by Manturov [7], invariants for classical knots are extended to invariants for virtual knots. In [3] Fedoseev and Manturov studied classical braids with structures, which are related to parity and dots on strands of classical braids. With those structures, parity can be extended to the case of braids and generalized parity can be deduced by braids with dots. The group of braids with parity and with points are defined as follows:

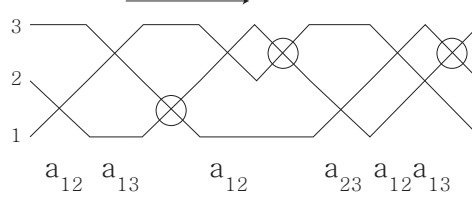


FIGURE 2. Example of a word for a free braid

Definition 2.5. [3] A group of n -braids with parity Br_n^2 is a group generated by $\{\sigma_i^0, \sigma_i^1\}_{i=1}^{n-1}$ with the following relations;

- (1) $\sigma_i^{\epsilon_1} \sigma_j^{\epsilon_2} = \sigma_j^{\epsilon_2} \sigma_i^{\epsilon_1}$ for $|i - j| > 2$,
- (2) $\sigma_i^{\epsilon_1} \sigma_{i+1}^{\epsilon_2} \sigma_i^{\epsilon_3} = \sigma_{i+1}^{\epsilon_3} \sigma_i^{\epsilon_2} \sigma_{i+1}^{\epsilon_1}$, where $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \pmod{2}$.

An element in Br_2^n is called a \mathbb{Z}_2 -braid.

Definition 2.6. [3] A group of n -braids with dots Br_n^2 is a group generated by $\{\sigma_i \mid i \in \{1, \dots, n-1\}\}, \{\tau_i \mid i \in \{1, \dots, n\}\}$ with the following relations;

- (1) $\sigma_i^2 = 1$ for all $i \in \{1, \dots, n-1\}$,
- (2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \geq 2$,
- (3) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n-1$,
- (4) $\tau_i^2 = 1$ for all $i \in \{1, \dots, n\}$,
- (5) $\tau_i \tau_j = \tau_j \tau_i$ for all $i, j \in \{1, \dots, n\}$,
- (6) $\tau_i \tau_{i+1} \sigma_i \tau_{i+1} \tau_i = \sigma_i$ for all $i \in \{1, \dots, n-1\}$,
- (7) $\sigma_i \tau_k = \tau_k \sigma_i$ for $|i - k| \geq 2$.

In the group Br_n^d , each generator τ_i is presented by a point on i -th strand of the braid. Now we define a mapping $f : Br_n^2 \rightarrow Br_n^d$ by

$$f(\sigma_i^\epsilon) = \begin{cases} \sigma_i & \text{if } \epsilon = 0, \\ \tau_i \sigma_i \tau_i & \text{if } \epsilon = 1. \end{cases}$$

In [3] they proved that this mapping is injective homomorphism. Moreover the following proposition can be proved.

Proposition 2.7. If two \mathbb{Z}_2 -braids are equal as braids with dots, they are equal as \mathbb{Z}_2 -braids.

In the following sections, we study G_n^2 with structures which are related to parity and points algebraically.

3. G_n^2 WITH ADDITIONAL STRUCTURES

In this section, we get enhancements of G_n^2 by adding some structures, which are related to parity and points on strands of colored pure braids. Firstly, we define a group $G_{n,p}^2$ and we call it G_n^2 with parity.

Definition 3.1. For a positive integer n , define $G_{n,p}^2$ by the group presentation generated by $\{a_{ij}^\epsilon \mid \{i, j\} \subset \{1, \dots, n\}, i < j, \epsilon \in \{0, 1\}\}$ with the following relations;

- (1) $(a_{ij}^\epsilon)^2 = 1$, $\epsilon \in \{0, 1\}$ and $i, j \in \{1, \dots, n\}$,
- (2) $a_{ij}^{\epsilon_{ij}} a_{kl}^{\epsilon_{kl}} = a_{kl}^{\epsilon_{kl}} a_{ij}^{\epsilon_{ij}}$ for $1 \leq i < j < k < l \leq n$,
- (3) $a_{ij}^{\epsilon_{ij}} a_{ik}^{\epsilon_{ik}} a_{jk}^{\epsilon_{jk}} = a_{jk}^{\epsilon_{jk}} a_{ik}^{\epsilon_{ik}} a_{ij}^{\epsilon_{ij}}$, for $1 \leq i < j < k \leq n$, where $\epsilon_{ij} + \epsilon_{ik} + \epsilon_{jk} = 0 \pmod{2}$.

The relations of $G_{n,p}^2$ are closely related to the axioms of parity [7]. We call $a_{ij}^0(a_{ij}^1)$ an *even generator* (an *odd generator*). If a word β in $G_{n,p}^2$ has no odd generators, then we call β an *even word* (or an *even element*). G_n^2 can be considered as a subgroup of $G_{n,p}^2$ by a homomorphism i from G_n^2 to $G_{n,p}^2$ defined by $i(a_{ij}) = a_{ij}^0$. It is easy to show that i is well-defined. Moreover, the following statement can be proved.

Lemma 3.2. *The homomorphism $i : G_n^2 \rightarrow G_{n,p}^2$ is a monomorphism.*

Proof. To prove this statement, consider projection map $p : G_{n,p}^2 \rightarrow G_n^2$ defined by

$$p(a_{ij}^\epsilon) = \begin{cases} a_{ij} & \text{if } \epsilon = 0, \\ 1 & \text{if } \epsilon = 1. \end{cases}$$

It is easy to show that p is well-defined function. Let β and β' be two words in G_n^2 such that $i(\beta) = i(\beta')$. By the definition of i , $p(i(\beta)) = \beta$ and $p(i(\beta')) = \beta'$. That is, $\beta = \beta'$ in G_n^2 , therefore, the proof is completed. \square

Corollary 3.3. *If two words β and β' in G_n^2 are equivalent in $G_{n,p}^2$, then they are equivalent in G_n^2 .*

Now, we define a group $G_{n,d}^2$ and we call it G_n^2 with points.

Definition 3.4. For a positive integer n , define $G_{n,d}^2$ by the group presentation generated by $\{a_{ij} \mid \{i, j\} \subset \{1, \dots, n\}, i < j\}$ and $\{\tau_i \mid i \in \{1, \dots, n\}\}$ with the following relations;

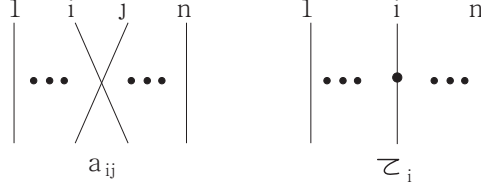
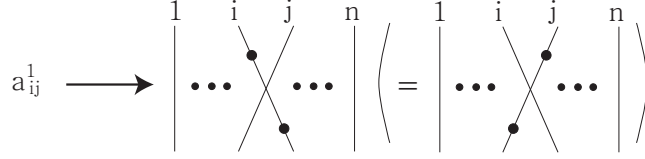
- (1) $a_{ij}^2 = 1$ for $\{i, j\} \subset \{1, \dots, n\}, i < j$,
- (2) $a_{ij} a_{kl} = a_{kl} a_{ij}$ for distinct $i, j, k, l \in \{1, \dots, n\}$,
- (3) $a_{ij} a_{ik} a_{jk} = a_{jk} a_{ik} a_{ij}$ for distinct $i, j, k \in \{1, \dots, n\}$
- (4) $\tau_i^2 = 1$ for $i \in \{1, \dots, n\}$,
- (5) $\tau_i \tau_j = \tau_j \tau_i$ for $i, j \in \{1, \dots, n\}$,
- (6) $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$ for $i, j \in \{1, \dots, n\}$,
- (7) $a_{ij} \tau_k = \tau_k a_{ij}$ for distinct $i, j, k \in \{1, \dots, n\}$.

We call τ_i a *generator for a point on i -th component* or simply, a *point on i -th component*. Geometrically, a_{ij} is corresponded to a 4-valent vertex and τ_i is corresponded to a point on the i -th strand of the free braid, see Fig. 3.

In fact, two groups $G_{n,p}^2$ and $G_{n,d}^2$ are closely related to each other. Define a function ϕ from $G_{n,p}^2$ to $G_{n,d}^2$ by

$$\phi(a_{ij}^\epsilon) = \begin{cases} a_{ij} & \text{if } \epsilon = 0, \\ \tau_i a_{ij} \tau_i & \text{if } \epsilon = 1. \end{cases}$$

Geometrically, the image of a_{ij}^1 is a crossing between i -th and j -th strands of the braid with two points just before and after of the crossing on i -th strand, see Fig. 4. Notice that, since we can get $\tau_i a_{ij} \tau_i = \tau_j a_{ij} \tau_j$ from the relation $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$, two diagrams with point in Fig. 4 are equivalent. On the other hand, the number

FIGURE 3. Geometrical meanings of a_{ij} and τ_i FIGURE 4. Image of a_{ij}^1

of τ_i and τ_j before a given crossing $\phi(a_{ij}^\epsilon)$ is equal to ϵ modulo 2, because every image of a_{kl}^ϵ before $\phi(a_{ij}^\epsilon)$ has two points or no points.

Lemma 3.5. *The map ϕ from $G_{n,p}^2$ to $G_{n,d}^2$ is well defined.*

Proof. To show that ϕ is well defined, it is enough to show that every relation is preserved by ϕ . For relations $(a_{ij}^\epsilon)^2 = 1$, if $\epsilon = 0$, then $\phi((a_{ij}^0)^2) = a_{ij}^2 = 1$. If $\epsilon = 1$, then

$$\phi((a_{ij}^1)^2) = \tau_i a_{ij} \tau_i \tau_i a_{ij} \tau_i = \tau_i a_{ij} a_{ij} \tau_i = \tau_i \tau_i = 1,$$

which can be presented geometrically as Fig. 5. For relations $a_{ij}^{\epsilon_{ij}} a_{kl}^{\epsilon_{kl}} = a_{kl}^{\epsilon_{kl}} a_{ij}^{\epsilon_{ij}}$ with distinct i, j, k, l , since i, j, k , and l are different indices, clearly the commutativity holds. For relations $a_{ij}^{\epsilon_{ij}} a_{ik}^{\epsilon_{ik}} a_{jk}^{\epsilon_{jk}} = a_{jk}^{\epsilon_{jk}} a_{ik}^{\epsilon_{ik}} a_{ij}^{\epsilon_{ij}}$, where $\epsilon_{ij} + \epsilon_{ik} + \epsilon_{jk} = 0 \pmod 2$, there are only two cases: all ϵ 's are equal to 0 or only two of them are equal to 1. If every ϵ is 0, then

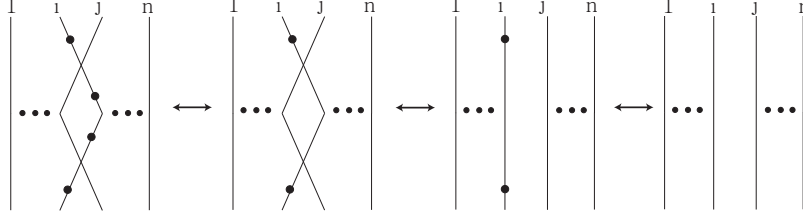
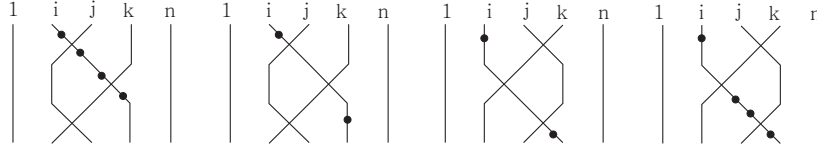
$$\phi(a_{ij}^0 a_{ik}^0 a_{jk}^0) = a_{ij} a_{ik} a_{jk} = a_{jk} a_{ik} a_{ij} = \phi(a_{jk}^0 a_{ik}^0 a_{ij}^0).$$

Suppose that only two of them are equal to 1, say $\epsilon_{ij} = 1$, $\epsilon_{ik} = 1$ and $\epsilon_{jk} = 0$. Then

$$\begin{aligned} \phi(a_{ij}^1 a_{ik}^1 a_{jk}^0) &= \tau_i a_{ij} \tau_i \tau_i a_{ik} \tau_i a_{jk} = \tau_i a_{ij} a_{ik} \tau_i a_{jk} = \tau_i a_{ij} a_{ik} a_{jk} \tau_i = \tau_i a_{jk} a_{ik} a_{ij} \tau_i = \\ &= \tau_i a_{jk} \tau_i \tau_i a_{ik} \tau_i a_{ij} = \phi(a_{jk}^0 a_{ik}^1 a_{ij}^1), \end{aligned}$$

which can be presented geometrically as Fig. 6.

Therefore, ϕ is well-defined.

FIGURE 5. $\phi(a_{ij}^2) = 1$ FIGURE 6. $\phi(a_{ij}^1 a_{ik}^1 a_{jk}^0) = \phi(a_{jk}^1 a_{ik}^1 a_{ij}^0)$

□

Let $H_{n,d}^2 = \{\beta \in G_{n,d}^2 \mid N_i(\beta) = 0 \bmod 2, i \in \{1, \dots, n\}\}$, where $N_i(\beta)$ is the number of τ_i in β . Notice that, by the definition of ϕ , it is clear that the image of $G_{n,p}^2$ by ϕ goes into the set $H_{n,d}^2$. Since every relation of $G_{n,d}^2$ preserves the number of τ 's modulo 2, $H_{n,d}^2$ is a subgroup of $G_{n,d}^2$.

Lemma 3.6. *Let $\beta = T_0 a_{i_1 j_1} T_2 \cdots T_{k-1} a_{i_k j_k} T_k \cdots T_{m-1} a_{i_m j_m} T_m \in H$, where T_k is a product of τ 's. Define a function $\chi : H_{n,d}^2 \rightarrow G_{n,p}^2$ by*

$$\chi(\beta) = a_{i_1 j_1}^{\epsilon_1} \cdots a_{i_k j_k}^{\epsilon_k} \cdots a_{i_m j_m}^{\epsilon_m},$$

where N_{i_k} is the number of τ_{i_k} in T_0, \dots, T_{k-1} and $\epsilon_k = N_{i_k} + N_{j_k}$. Then χ is well-defined.

Proof. Consider $\beta = T_0 a_{i_1 j_1} T_2 \cdots T_{k-1} a_{i_k j_k} T_k \cdots T_{m-1} a_{i_m j_m} T_m \in H$. Assume that $a_{i_k j_k}$ contains in one of relations, which can be applied to β . If $a_{i_k j_k}$ contains in the relation (6) $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$, then the number of τ_{i_k} and τ_{j_k} in T_{k-1} preserves modulo 2 and then the sum of the numbers of τ_{i_k} and τ_{j_k} in T_0, \dots, T_{k-1} is not changed modulo 2. Therefore $\epsilon_k = N_{i_k} + N_{j_k}$ is preserved modulo 2. If $a_{i_k j_k}$ is contained in the relation (7) $\tau_k a_{ij} = a_{ij} \tau_k$, then the number of τ_{i_k} and τ_{j_k} in T_{k-1} is preserved, since in relation (7) i, j, k are different respectively. Therefore the number of τ_{i_k} and τ_{j_k} in T_0, \dots, T_{k-1} is not changed, and $\epsilon_k = N_{i_k} + N_{j_k}$ is preserved modulo 2. The relation (2) $a_{ij} a_{kl} = a_{kl} a_{ij}$ is preserved along χ by relation $a_{ij}^{\epsilon_1} a_{kl}^{\epsilon_2} = a_{kl}^{\epsilon_2} a_{ij}^{\epsilon_1}$

for different i, j, k, l . Suppose that $a_{i_k j_k}$ is appeared in the relation (1) $a_{ij}^2 = 1$. Then $i_k = i_{k+1}$, $j_k = j_{k+1}$ and there are no τ 's in T_k . Therefore $\epsilon_k = \epsilon_{k+1}$ and the relation is preserved by $(a_{ij}^\epsilon)^2 = 1$. Suppose that $a_{i_k j_k}$ is appeared in relation (3) $a_{ij} a_{ik} a_{jk} = a_{jk} a_{ik} a_{ij}$. Suppose that $a_{i_k j_k} a_{i_{k+1} j_{k+1}} a_{i_{k+2} j_{k+2}} = a_{i_{k+2} j_{k+2}} a_{i_{k+1} j_{k+1}} a_{i_k j_k}$ and $i_k = i_{k+1}$, $j_k = i_{k+2}$ and $j_{k+1} = j_{k+2}$. It is sufficient to show that $\epsilon_k + \epsilon_{k+1} + \epsilon_{k+2} = 0$ modulo 2. Then there are no τ 's in T_k, T_{k+1}, T_{k+2} , and hence $N_{i_k} = N_{i_{k+1}}, N_{j_k} = N_{i_{k+2}}, N_{j_{k+1}} = N_{j_{k+2}}$. Therefore

$$\epsilon_k + \epsilon_{k+1} + \epsilon_{k+2} = N_{i_k} + N_{j_k} + N_{i_{k+1}} + N_{j_{k+1}} + N_{i_{k+2}} + N_{j_{k+2}} = 0 \pmod{2}.$$

and the proof is completed. \square

Theorem 3.7. $G_{n,p}^2$ is isomorphic to $H_{n,d}^2$.

Proof. We will show that $\chi \circ \phi = 1_{G_{n,p}^2}$ and $\phi \circ \chi = 1_{H_{n,d}^2}$. To show that $\chi \circ \phi = 1_{G_{n,p}^2}$ let $\beta = a_{i_1 j_1}^{\epsilon_1} \cdots a_{i_m j_m}^{\epsilon_m} \in G_{n,p}^2$. Then

$$\chi(\phi(\beta)) = \chi(\tau_{i_1}^{\epsilon_1} a_{i_1 j_1} \tau_{i_1}^{\epsilon_1} \cdots \tau_{i_m}^{\epsilon_m} a_{i_m j_m} \tau_{i_m}^{\epsilon_m}) = a_{i_1 j_1}^{\theta_1} \cdots a_{i_m j_m}^{\theta_m}.$$

For each k , the number of all τ 's which are appeared from $a_{i_1 j_1}^{\epsilon_1} \cdots a_{i_m j_m}^{\epsilon_m}$ is even. Since θ_k is equal to the number of all τ_{i_k} and τ_{j_k} modulo 2, $\theta_k = \epsilon_k$. Therefore $\chi \circ \phi = 1_{G_{n,p}^2}$.

Now we will show that $\phi \circ \chi = 1_{H_{n,d}^2}$. Let

$$\beta = T_0 a_{i_1 j_1} T_1 \cdots T_{k-1} a_{i_k j_k} T_k \cdots T_{m-1} a_{i_m j_m} T_m \in H.$$

Firstly we will show that β is equivalent to an element in the form

$$\tau_{i_1}^{\epsilon_1} a_{i_1 j_1} \tau_{i_1}^{\epsilon_1} \cdots \tau_{i_m}^{\epsilon_m} a_{i_m j_m} \tau_{i_m}^{\epsilon_m}.$$

By the relations $\tau_i \tau_j = \tau_j \tau_i$ and $\tau_k a_{ij} = a_{ij} \tau_k$, β is equivalent to the form

$$T_0 a_{i_1 j_1} T_2 \cdots T_{k-1} a_{i_k j_k} T_k \cdots T_{m-1} a_{i_m j_m} T_m$$

such that T_k only has $\tau_{i_k}, \tau_{j_k}, \tau_{i_{k+1}}$ and $\tau_{j_{k+1}}$. In point of $a_{i_k j_k}$, it is a product of the following;

$$A_{i_k j_k} = \tau_{i_k}^{\theta_{i_k}^f} \tau_{j_k}^{\theta_{j_k}^f} a_{i_k j_k} \tau_{i_k}^{\theta_{i_k}^b} \tau_{j_k}^{\theta_{j_k}^b}.$$

Now we claim that there is an element β' which is a product of

$$A'_{i_k j_k} = \tau_{i_k}^{\theta_{i_k}^f} \tau_{j_k}^{\theta_{j_k}^f} a_{i_k j_k} \tau_{i_k}^{\theta_{i_k}^b} \tau_{j_k}^{\theta_{j_k}^b},$$

where $\tau_{i_k}^{\theta_{i_k}^f} = \tau_{i_k}^{\theta_{i_k}^b}$ and $\tau_{j_k}^{\theta_{j_k}^f} = \tau_{j_k}^{\theta_{j_k}^b}$. Suppose that i is an index such that $A_{ij} = \tau_i^{\theta_i^f} \tau_j^{\theta_j^f} a_{ij} \tau_i^{\theta_i^b} \tau_j^{\theta_j^b}$ is the first part in which $\theta_i^f \neq \theta_i^b$, say $\theta_i^f = 1$ and $\theta_i^b = 0$. Notice that if $\theta_i^b = 1$ and $\theta_i^f = 1$, then $\tau_i^{\theta_i^b}$ can be move to the very next A_{il} and it is same to the case of $\theta_i^f = 1$ and $\theta_i^b = 0$. Since β is in H , the number of τ_i should be even and there is $A_{ik} = \tau_i^{\theta_i^f} \tau_k^{\theta_k^f} a_{ik} \tau_i^{\theta_i^b} \tau_k^{\theta_k^b}$ such that $\theta_i^f = 1$ or $\theta_i^b = 1$ as 3.1 and 3.2.

$$(3.1) \quad \beta = \cdots A_{ij} \cdots A_{ik} \cdots$$

$$(3.2) \quad = \cdots \tau_j^{\theta_j^f} a_{ij} \tau_i^{\theta_i^b} \tau_j^{\theta_j^b} \cdots \tau_i^{\theta_i^f} \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots.$$

If $\theta_i'^f = 1$ and there are no A_{il} , then by the relations $a_{ij}\tau_k = \tau_k a_{ij}$ and $\tau_i\tau_j = \tau_j\tau_i$, τ_i can be move to the next of a_{ij} as 3.3 and 3.4.

$$(3.3) \quad \beta = \cdots \tau_i \tau_j^{\theta_j^f} a_{ij} \tau_j^{\theta_j^b} \cdots A_{**} \cdots \tau_i \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots$$

$$(3.4) \quad = \cdots \tau_i \tau_j^{\theta_j^f} a_{ij} \tau_i \tau_j^{\theta_j^b} \cdots A_{**} \cdots \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots$$

If $\theta_i'^f = 1$ and there is a A_{il} 's, then by the relation $\tau_i^2 = 1$ new τ 's are appeared and by the relations $a_{ij}\tau_k = \tau_k a_{ij}$ and $\tau_i\tau_j = \tau_j\tau_i$, τ_i 's can be move to the next of a_{ij} and to the front of a_{il} as 3.5, 3.6 and 3.7.

$$(3.5) \quad \beta = \cdots \tau_i \tau_j^{\theta_j^f} a_{ij} \tau_j^{\theta_j^b} \cdots A_{il} \cdots \tau_i \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots$$

$$(3.6) \quad = \cdots \tau_i \tau_j^{\theta_j^f} a_{ij} \tau_j^{\theta_j^b} \cdots \tau_i \tau_i A_{il} \cdots \tau_i \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots$$

$$(3.7) \quad = \cdots \tau_i \tau_j^{\theta_j^f} a_{ij} \tau_i \tau_j^{\theta_j^b} \cdots \tau_i A_{il} \tau_i \cdots \tau_k^{\theta_k^f} a_{ik} \tau_k^{\theta_k^b} \cdots$$

Analogously the case $\theta_i'^b = 1$ also can be solved.

Now assume that $\beta = \prod_{k=1}^m \tau_{i_k}^{\theta_{i_k}} \tau_{j_k}^{\theta_{j_k}} a_{i_k j_k} \tau_{i_k}^{\theta_{i_k}} \tau_{j_k}^{\theta_{j_k}}$. Then

$$\begin{aligned} \phi \circ \chi(\beta) &= \phi \circ \chi \left(\prod_{k=1}^m \tau_{i_k}^{\theta_{i_k}} \tau_{j_k}^{\theta_{j_k}} a_{i_k j_k} \tau_{i_k}^{\theta_{i_k}} \tau_{j_k}^{\theta_{j_k}} \right) \\ &= \phi \left(\prod_{k=1}^m a_{i_k j_k}^{\theta_{i_k} + \theta_{j_k}} \right) = \prod_{k=1}^m (\tau_{i_k}^{\theta_{i_k} + \theta_{j_k}} a_{i_k j_k} \tau_{i_k}^{\theta_{i_k} + \theta_{j_k}}), \end{aligned}$$

where $\theta_{i_k j_k} = \theta_{i_k} + \theta_{j_k} \bmod 2$. If $\theta_{i_k} = \theta_{j_k} = 1$, by the relation $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$, β and $\phi \circ \chi(\beta)$ are same elements in $G_{n,d}^2$. If $\theta_{i_k} = 1$ and $\theta_{j_k} = 0$ (or $\theta_{i_k} = 0$ and $\theta_{j_k} = 1$), by the relation $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$ (in other word, $\tau_i a_{ij} \tau_i = \tau_j a_{ij} \tau_j$), β and $\phi \circ \chi(\beta)$ are same elements in $G_{n,d}^2$ and hence $\phi \circ \chi = 1_{G_{n,d}^2}$. \square

4. $G_{n,d}^2$ AND G_{n+1}^2

In [8], maps from G_n^k to G_{n-1}^k and from G_n^k to G_{n-1}^{k-1} are constructed. In these maps, each generator is mapped to a generator. Here we justify these maps and extended to the case of mappings from G_n^2 to $G_{n-1,d}^2$. In section 3, we obtained including maps $i : G_n^2 \rightarrow G_{n,p}^2$ and $\phi : G_{n,p}^2 \rightarrow G_{n,d}^2$. To show that they are including maps, we found right inverse of them. In this section we consider relations between $G_{n,d}^2$ and G_{n+1}^2 .

Let us define ψ from G_{n+1}^2 to $G_{n,d}^2$ by

$$\psi(a_{ij}) = \begin{cases} a_{ij} & \text{if } n+1 \notin \{i, j\}, \\ \tau_i & \text{if } j = n+1, \\ \tau_j & \text{if } i = n+1. \end{cases}$$

Then the following lemma can be proved.

Lemma 4.1. *A mapping ψ from G_{n+1}^2 to $G_{n,d}^2$ is well-defined.*

Proof. It is enough to show that every relation for G_{n+1}^2 is preserved by ψ . If every index is different with $n+1$, it is clear. Consider a relation $a_{ij}a_{kl} = a_{kl}a_{ij}$ for distinct $i, j, k, l \in \{1, \dots, n+1\}$. If one of i, j, k and l is $n+1$, say $i = n+1$, then

$$\psi(a_{(n+1)j}a_{kl}) = \tau_j a_{kl} = a_{kl} \tau_j = \psi(a_{kl}a_{(n+1)j}).$$

For the relations $a_{ij}a_{ik}a_{jk} = a_{jk}a_{ik}a_{ij}$, if one of i, j and k is $n+1$, say $i = n+1$, then

$$\psi(a_{ij}a_{ik}a_{jk}) = \tau_j \tau_k a_{jk} = a_{jk} \tau_k \tau_j = \psi(a_{jk}a_{ik}a_{ij}).$$

Clearly, the relation $a_{ij}^2 = 1$ is preserved along ψ and the statement is proved. \square

Now we define inverse mapping. We define ω from $G_{n,d}^2$ to G_{n+1}^2 by $\omega(a_{ij}) = a_{ij}$ and $\omega(\tau_i) = a_{i(n+1)}$, for example, see Fig. 7.

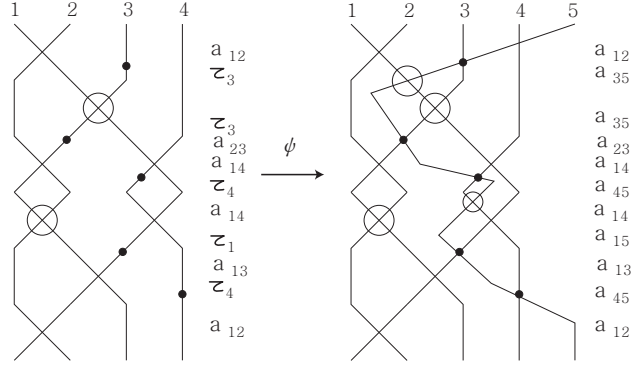


FIGURE 7. Example of a image of the mapping ω

But this mapping is not well-defined because the relation $\tau_i \tau_j = \tau_j \tau_i$ in $G_{n,d}^2$ goes to $a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)}$ but $a_{i(n+1)}$ and $a_{j(n+1)}$ are not commutative. However, we can modify the mapping and get the following lemma.

Lemma 4.2. *A mapping $\omega : G_{n,d}^2 \rightarrow G_{n+1}^2 / \langle a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)} \rangle$, defined by $\omega(a_{ij}) = a_{ij}$ and $\tau_i = a_{i(n+1)}$, is well-defined.*

Proof. It is sufficient to show that every relation for $G_{n,d}^2$ goes to trivial in $G_{n+1}^2 / \langle a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)} \rangle$. For relations (1),(2),(3) and (4), it is clear. Consider relation $\tau_i \tau_j = \tau_j \tau_i$. Then

$$\omega(\tau_i \tau_j) = a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)} = \omega(\tau_j \tau_i).$$

The relation (6) $\tau_i \tau_j a_{ij} \tau_j \tau_i = a_{ij}$ can be written by $\tau_i \tau_j a_{ij} = a_{ij} \tau_j \tau_i$. Then

$$\omega(\tau_i \tau_j a_{ij}) = a_{i(n+1)}a_{jn+2}a_{ij} = a_{ij}a_{j(n+1)}a_{i(n+1)} = \omega(a_{ij} \tau_j \tau_i).$$

Finally, the relations (5) and (7) are preserved by the relation $a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)}$ and the proof is completed. \square

Geometrically the relations $a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)}$ mean that two consecutive classical crossings pass a virtual crossing, see Fig. 8. Generally, this move in this case, is called a *forbidden move by $(n+1)$ -th strand*.

Corollary 4.3. *The groups $G_{n,d}^2$ and $G_{n+1}^2 / \langle a_{i(n+1)}a_{j(n+1)} = a_{j(n+1)}a_{i(n+1)} \rangle$ are isomorphic.*

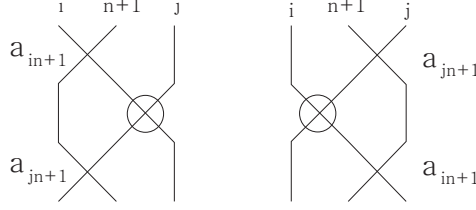


FIGURE 8. forbidden moves

Proof. We will show that $\omega \circ \psi = 1_{G_{n,d}^2}$ and $\psi \circ \omega = 1_{G_{n+1}^2 / \langle a_{i(n+1)} a_{j(n+1)} = a_{j(n+1)} a_{i(n+1)} \rangle}$. For generators a_{ij} in $G_{n+1}^2 / \langle a_{i(n+1)} a_{j(n+1)} = a_{j(n+1)} a_{i(n+1)} \rangle$,

$$\psi(a_{ij}) = \begin{cases} a_{ij} & \text{if } n+1 \in \{i, j\}, \\ \tau_i & \text{if } j = n+1, \\ \tau_j & \text{if } i = n+1. \end{cases}$$

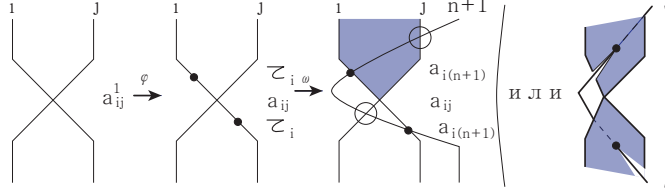
By definition of ω , $\omega(\psi(a_{ij})) = a_{ij}$. Hence $\omega \circ \psi = 1_{G_{n,d}^2}$. Analogously, it can be shown that $\psi \circ \omega = 1_{G_{n+1}^2 / \langle a_{i(n+1)} a_{j(n+1)} = a_{j(n+1)} a_{i(n+1)} \rangle}$. \square

We proved that there are isomorphism from $G_{n,p}^2$ to $H_{n,d}^2 \subset G_{n,d}^2$ (Theorem ??) and isomorphism from $G_{n,d}^2$ to $G_{n+1}^2 / \langle a_{i(n+1)} a_{j(n+1)} = a_{j(n+1)} a_{i(n+1)} \rangle$ (Theorem ??). From those theorems, it is followed that there is a monomorphism $\omega \circ \phi$ from $G_{n,p}^2$ to $G_{n+1}^2 / \langle a_{i(n+1)} a_{j(n+1)} = a_{j(n+1)} a_{i(n+1)} \rangle$. It means that every braid of n strands with parity can be presented by braid with $(n+1)$ strands up to forbidden moves by $(n+1)$ -th strand. That is, braids with parity are in itself. On the other hand, parity of braid has geometrical meaning: how $(n+1)$ -th strand is knotted with area, which is surrounded by other strands, see Fig. 9. In Fig. 9, the image of a_{ij}^1 by $\omega \circ \phi$ $(n+1)$ -th strand passes area in black, which is surrounded by i -th and j -th strands. Notice that the image of a_{ij}^1 by $\omega \circ \phi$ any relation in G_{n+1}^2 cannot be applied. That is, linking between $(n+1)$ -th strand and the area in black cannot be disappeared.

Conversely, for a braid β with $(n+1)$ strands such that the number of crossings between i -th and $(n+1)$ -th strands is even for every index $i \neq n+1$, that is, a braid of n strands with parity (or with points) can be obtained from $\beta \in \psi^{-1}(H_{n,d}^2)$ by deleting one strand. Now define a mapping $\psi_m : G_{n+1}^2 \rightarrow G_{n,d}^2$ by

$$\psi_m(a_{ij}) = \begin{cases} \tau_i & \text{if } j = n+1, i < m, \\ \tau_{i-1} & \text{if } j = n+1, i > m. \\ a_{ij} & \text{if } i, j < m, i, j \neq n+1 \\ a_{i(j-1)} & \text{if } i < m, j > m, i, j \neq n+1 \\ a_{(i-1)(j-1)} & \text{if } i, j > m, i, j \neq n+1 \end{cases}$$

Analogously ψ_m is well-defined (see lemma 4.1). Then, for each braid β of $(n+1)$ strands such that the number of crossings between i -th and m -th strands is even

FIGURE 9. The image of a_{ij}^1 by $\omega \circ \phi$

for each index $i \neq m$, that is, $\beta \in \psi_m^{-1}(H_{n,d}^2)$, and a braid with parity can be obtained.

Example 4.4. Let $\beta = a_{12}a_{23}a_{13}a_{23}a_{13}a_{23}a_{12}a_{23}$ in G_3^2 . Note that $\psi_m(\beta) \in H_{2,d}^2$ for each index m . Then

- $\chi(\psi_1(\beta)) = \chi(\tau_1 a_{12} \tau_2 a_{12} \tau_2 a_{12} \tau_1 a_{12}) = a_{12}^1 a_{12}^0 a_{12}^1 a_{12}^0 \neq 1$,
- $\chi(\psi_2(\beta)) = \chi(\tau_1 \tau_2 a_{12} \tau_2 a_{12} \tau_2 \tau_1 \tau_2) = \chi(a_{12} \tau_1 a_{12} \tau_1) = a_{12}^0 a_{12}^1 \neq 1$,
- $\chi(\psi_3(\beta)) = \chi(a_{12} \tau_2 \tau_1 \tau_2 \tau_1 \tau_2 a_{12} \tau_2) = \chi(a_{12} \tau_2 a_{12} \tau_2) = a_{12}^0 a_{12}^1 \neq 1$.

Since ψ_m is well defined, β is not trivial in G_3^2 .

As the above example, for given brad β in $\psi_m^{-1}(H_{n,d}^2) \subset G_{n+1}^2$, if $\chi(\psi_m(\beta)) \neq 1$, then β is not trivial and m -th strand of β is important to be non-trivial for β . And the following question is followed:

Assume that β in $\psi_m^{-1}(H_{n,d}^2) \subset G_{n+1}^2$ for each index m . If β it non-trivial in G_{n+1}^2 , then is there index m such that $\chi(\psi_m(\beta)) \neq 1$?

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